Homogeneous equations

A system of equations is <u>homogeneous</u> if all the constant terms are zero. i.e. each equation has the form $a_1 x_1 + \cdots + a_n x_n = 0$.

$x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is always a solution to a homog. system, called the <u>trivial solution</u> all other solutions are <u>nontrivial solutions</u>.

Ex:

$$x_1 + x_2 - x_4 = 0$$
$$2x_1 - x_2 + 3x_3 + x_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 2 & -1 & 3 & 1 & 0 \end{array}\right] \xrightarrow{②-2①} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & -3 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}②} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right] \xrightarrow{①-②} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right]$$

Solve for leading variables x_1, x_2:

$$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$$

$$x_2 - x_3 - x_4 = 0 \Rightarrow x_2 = x_3 + x_4.$$

Set $x_3 = s$, $x_4 = t$.

Parametric solution:

$$\boxed{\begin{aligned} x_1 &= -s \\ x_2 &= s+t \\ x_3 &= s \\ x_4 &= t \end{aligned}}$$

We can get a nontrivial solution by (for example) setting $s=1$, $t=2$.
Then $x_1=-1$, $x_2=3$, $x_3=1$, $x_4=2$ is a nontrivial solution.

Note that there is a nontrivial solution because there are infinitely many solutions, i.e. there are parameters in the solution. This is because there were nonleading variables, since there are more variables than equations.
More generally:

Theorem: If a homog. system has more variables than equations, it will have a nontrivial solution (infinitely many!).

Note that the converse doesn't hold:

$$\begin{aligned} 2x+2y&=0 \\ x+y&=0 \end{aligned}$$ has 2 variables, 2 equations, but

$x=-1$, $y=1$ is a nontrivial solution.

Linear combinations

Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be two columns, or vectors (more on these later). We can add them and take scalar multiples as follows:

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$ is the sum of $\vec{x}$ and $\vec{y}$.

If k is a number (not a vector), then

$$k\vec{x} = \begin{bmatrix} kx_1 \\ \vdots \\ kx_n \end{bmatrix}$$ is the scalar product, or a scalar multiple of $\vec{x}$.

A sum of scalar multiples of several columns is a linear combination of the columns.

Ex: If $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, then

$$3\vec{x} - 2\vec{y} = \begin{bmatrix} 3 \\ 6 \\ 15 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 15 \end{bmatrix}$$

is a linear combination of $\vec{x}$ and $\vec{y}$.

Ex: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

The third entry of any such linear combination would be 0.

Ex: Is $\begin{bmatrix}1\\1\\1\end{bmatrix}$ a linear combination of $\underbrace{\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{=\vec{x}}, \underbrace{\begin{bmatrix}0\\2\\3\end{bmatrix}}_{=\vec{y}}, \underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}}_{=\vec{z}}$?

i.e. can we find numbers a, b, c s.t.

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = a\vec{x} + b\vec{y} + c\vec{z} = \begin{bmatrix}a\\-a\\0\end{bmatrix} + \begin{bmatrix}0\\2b\\3b\end{bmatrix} + \begin{bmatrix}c\\c\\2c\end{bmatrix}$$

$$= \begin{bmatrix}a+c\\-a+2b+c\\3b+2c\end{bmatrix}$$

This is now a system of equations w/ a,b,c as variables which we can denote w/ augmented matrix

$$\left[\begin{array}{ccc|c}1&0&1&1\\-1&2&1&1\\0&3&2&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&1&1\\0&2&2&2\\0&3&2&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&1&1\\0&1&1&1\\0&3&2&1\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&1&1\\0&1&1&1\\0&0&-1&-2\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&1&1\\0&1&1&1\\0&0&1&2\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&0&-1\\0&1&1&1\\0&0&1&2\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&-1\\0&0&1&2\end{array}\right]$$ So if we set $a=-1$, $b=-1$, $c=2$,

$\begin{bmatrix}1\\1\\1\end{bmatrix} = -\vec{x} - \vec{y} + 2\vec{z}$, so it <u>is</u> a linear combination.

Basic solutions to homogeneous systems

Linear combinations also help us express solutions to homogeneous systems.

Ex: In the example near the beginning of the section, the parametric solution to a homog. system was

$$\boxed{\begin{aligned} x_1 &= -s \\ x_2 &= s+t \\ x_3 &= s \\ x_4 &= t \end{aligned}}$$

Alternately, we can write the solutions in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s \\ s+t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

for any values of s and t. Note that if $s=1$, $t=0$, we get the solution

$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. If $s=0$, $t=1$, we get solution $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and

The rest of the solutions are exactly the linear combinations of these two. This is true more generally. i.e.:

Theorem: Any linear combination of solutions to a homogeneous system is again a solution.

In the above example, $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ are called basic solutions to the linear system. In general, the gaussian algorithm produces one basic solution for each parameter.

Ex: $\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$

$x_1 + 2x_2 - 2x_4 + 3x_5 = 0 \longrightarrow x_1 = -2x_2 + 2x_4 - 3x_5$

$x_3 + x_4 - x_5 = 0 \Longrightarrow x_3 = -x_4 + x_5$.

Set $x_2 = r$, $x_4 = s$, $x_5 = t$.

Solution:

$x_1 = -2r + 2s - 3t$

$x_2 = r$

$x_3 = -s + t$

$x_4 = s$

$x_5 = t$

In vector form:

$$\begin{bmatrix} -2r + 2s - 3t \\ r \\ -s + t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2s \\ 0 \\ -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ 0 \\ t \\ 0 \\ t \end{bmatrix}$$

$$= r \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + s \underbrace{\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}} + t \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

basic solutions

Note that once we know the basic solutions, we know all the solutions!

Practice problems: 1.3.2 ab, 1.3.4, 1.3.5 bd, 1.3.7, 1.3.9